Numerisch berechnet wurden der Imaginärteil, der dispersive Anteil des Realteils und der gesamte Realteil des optischen Potentials für verschiedene Werte des Fermi-Impulses p_F. Um die Anzahl der Integrationen zu beschränken, wurde teilweise die Raumwinkelmittlung nach BRUECKNER[12] verwendet. Die Ergebnisse sind in den Abb. 1 – 3 dargestellt. Es ergeben sich ähnliche Werte wie bei HODGSON[13] und in der Gasapproximation[14]. Den Hauptbeitrag zum Realteil des optischen Potentials liefert der Hartree-Fock-Anteil (s. Abb. 1 und 2). Verwendet man eine effektive Massenapproximation, so würde sowohl der Realteil als auch der Imaginärteil kleiner werden. Zu Vergleichszwecken sind in den Abb. 4 – 6 das optische Potential bei Beschränkung auf den S-Wellenanteil dargestellt. Der Imaginärteil und der Realteil werden wie in der Gasapproximation[14] und der Λ_{00}-Approximation[10] kleiner. Im Gegensatz zu SITENKO[8] beschränken wir uns bei den S-Wellen nicht auf ein rein attraktives Nukleon-Nukleon-Potential.

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[12] K. A. BRUECKNER, The Many Body Problem, Wiley, New York 1958.

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Kinetic Theory for a Dilute Gas of Particles with Spin. II. Relaxation Coefficients

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The relaxation coefficients to be discussed are given by collision brackets pertaining to the linearized collision operator of the generalized Boltzmann equation for particles with spin. The order of magnitude of various nondiagonal relaxation coefficients which are of interest for the SENFTLEBEN-BEENAKKER effect is investigated. Those nondiagonal relaxation coefficients which are linear in the nonsphericity parameter ε (ε essentially measures the ratio of the nonspherical and the spherical parts of the interaction potential), as well as some diagonal relaxation coefficients are expressed in terms of generalized Omega-integrals.

In the previous paper[1], by application of the moment method, a set of "transport — relaxation equations" (TRE) has been derived from the generalized Boltzmann equation for particles with spin[2]. The transport — relaxation equations comprise the constitutive laws (in first and second hydrodynamical approximations) for all the irreversible processes of interest. The pertaining transport constants can be expressed in terms of certain relaxation coefficients, i.e. collision brackets.

In view of the SENFTLEBEN-BEENAKKER effect[3,4] — influence of a magnetic field on the transport properties of polyatomic gases — the correlation between linear momentum and internal angular momentum ("spin") of molecules, set up in a transport situation, is of primary interest. This correlation — without field — causes the transport constants to differ by a certain small amount from the "isotropic" values which would be found without such correlation. However, if a magnetic field is applied, the correlation between linear and angular momenta is partially destroyed by the precessional motion of the "spins". Consequently, the transport constants, at least partially, approach their isotro-

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[1] S. HESS and L. WALDMANN, Z. Naturforsch. **21a**, 1529 [1966]. In the following, this paper will be referred to as I.

[2] L. WALDMANN, Z. Naturforsch. **12a**, 660 [1957]; **13a**, 609 [1958]. — R. F. SNIDER, J. Chem. Phys. **32**, 1051 [1960].

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pic values. This qualitatively explains the magnetic field dependence of heat conductivity and viscosity observed experimentally with paramagnetic[3] and with diamagnetic gases[4].

To get detailed insight into the nature of this correlation and its role in the Senftleben-Beenakker effect one has to study the relative order of magnitude of the various "nondiagonal" relaxation coefficients which couple — in the TRE — the heat flux vector and the friction pressure tensor with moments that are mean values of "spin"-dependent expansion vectors or tensors. In the present paper, such an investigation shall be given for particles with spin (in the proper sense).

As an example, ortho-hydrogen molecules at low temperatures (where only the rotational state $l=1$ is occupied) may be considered as such particles with spin in the proper sense. Although the model of spin-particles has a limited applicability for polyatomic gases, some of the results obtained with this model may be representative and typical for any molecule with rotational degrees of freedom.

The kinetic theory of the Senftleben-Beenakker effect has been treated within the last few years in a great number of papers starting with the one by Kagan and Maksimov[5], continued by Waldmann and Kupatt[6], McCourt and Snider[7], Knaap and Beenakker[8], and Tip[8a]. For a full list of literature, reference is made to Beenakker's recent review article[9].

Firstly, some general remarks on the collision brackets for particles with spin are made (§ 1). Then the diagonal relaxation coefficients pertaining to the heat flux and friction pressure are related to Ω-integrals where, instead of the scattering cross section of the monatomic gas, a spin averaged cross section appears (§ 2). In the main part of this paper (§ 3—5), we are concerned with the orders of magnitude of those nondiagonal relaxation coefficients which are of interest for the Senftleben-Beenakker effect on the heat conductivity and the viscosity. A scaling factor ε is introduced which essentially measures the ratio of the nonspherical and the spherical parts of the interaction potential. A nonspherical scattering amplitude operator is used involving spin-dependent terms of known orders of magnitude[10]. The nondiagonal relaxation coefficients of order ε are expressed in terms of generalized Ω-integrals. Next, in "spherical approximation" (i.e. the nonspherical part of the scattering amplitude is neglected) some diagonal relaxation coefficients for moments of spin dependent quantities are related to Ω-integrals involving the spherical part of the scattering cross section only (§ 6). Finally, we briefly compare the orders of magnitude of the nondiagonal relaxation coefficients given here for the model of spin particles with those known for the case where inelastic collisions are of importance (diatomic gas at room temperature) (§ 7).

§ 1. General Remarks on Collision Brackets

The non-equilibrium state of a gas consisting of particles with spin is described by the one particle distribution function

$$f_1 = f(t, \boldsymbol{x}, \boldsymbol{V}_1, \boldsymbol{s}_1) \tag{1.1}$$

with the normalization

$$\mathrm{tr}_1 \int \mathrm{d}^3 V_1 f_1 = n\,, \tag{1.2}$$

where n is the number density. Here $\boldsymbol{V}_1$ is the velocity of a particle measured in units of

$$\sqrt{\frac{2kT_0}{m}} = \sqrt{\frac{2}{3}}\, v_0\,, \tag{1.3}$$

(T_0: equilibrium temperature), $\boldsymbol{s}_1$ is the vector spin operator (in units of $\hbar$).

In I, the deviation of f_1 from its equilibrium value

$$f_{01} = n_0 \pi^{-3/2} e^{-V_1^2} \tag{1.4}$$

(n_0: equilibrium number density) has been expanded with respect to a complete set of orthonormalized Cartesian tensors $\Phi^{(i)}_{\mu_1\dots\mu_l}$ (parity $P=(-1)^l$) and pseudo tensors $\Psi^{(i)}_{\mu_1\dots\mu_l}$ ($P=(-1)^{l+1}$) of rank $l=0, 1, 2\dots$ depending on $\boldsymbol{V}_1$ and $\boldsymbol{s}_1$. Expansion tensors of equal rank and parity are distinguished by superscripts. The mean values of these expansion tensors

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[5] Y. Kagan and L. Maksimov, Sov. Phys. JETP **14**, 604 [1962].

[6] L. Waldmann and H. D. Kupatt, Z. Naturforsch. **18a**, 86 [1963].

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[10] S. Hess and W. E. Köhler, Z. Naturforsch. **23a**, 1903 [1968].

evaluated according to

$$\langle \Phi \rangle = \frac{1}{n} \operatorname{tr}_1 \int \mathrm{d}^3 V_1 f_1 \Phi(\boldsymbol{V}_1, \boldsymbol{s}_1) \tag{1.5}$$

are the time and space dependent expansion coefficients $a^{(i)}_{\mu_1 \dots \mu_l}$ and $b^{(i)}_{\mu_1 \dots \mu_l}$ which appear in the transport-relaxation equations (TRE). The relaxation coefficients $\omega^{(ik)}_{Pl}$ between the l-th rank tensors with parity P (also occurring in the TRE) are related to collision brackets in the following way:

$$\omega^{(ik)}_{Pl} = (2l+1)^{-1} \langle \Phi^{(i)}_{\mu_1 \dots \mu_l} \omega(\Phi^{(k)}_{\mu_1 \dots \mu_l}) \rangle_0 . \tag{1.6}$$

Here $\omega(\Phi)$ is the linearized collision operator of the transport equation for particles with spin. The bracket $\langle \quad \rangle_0$ refers to an average like (1.5) but with f_{01} instead of f_1. General properties of the collision brackets (parity and time reversal behavior, definiteness, symmetries) have been studied earlier[1].

For particles with spin the relative kinetic energy is not changed during a binary collision. Of course total momentum is conserved too. So, one has the encounter relations for a pair of particles labelled by 1 and 2 (where primed quantities refer to the precollisional state and unprimed ones to the post-collisional state):

$$\begin{aligned} \boldsymbol{V}_1 &= \frac{1}{\sqrt{2}}(\boldsymbol{V} + \gamma \boldsymbol{e}), \quad \boldsymbol{V}_1' = \frac{1}{\sqrt{2}}(\boldsymbol{V} + \gamma \boldsymbol{e}'); \\ \boldsymbol{V}_2 &= \frac{1}{\sqrt{2}}(\boldsymbol{V} - \gamma \boldsymbol{e}), \quad \boldsymbol{V}_2' = \frac{1}{\sqrt{2}}(\boldsymbol{V} - \gamma \boldsymbol{e}'). \end{aligned} \tag{1.7}$$

Here $\boldsymbol{V}$ is a dimensionless center of mass velocity of the pair of particles, γ the magnitude of the dimensionless relative velocity, $\boldsymbol{e}'$ and $\boldsymbol{e}$ are unit vectors in the direction of the relative velocity before and after collision:

$$\begin{aligned} \tfrac{1}{2}(\boldsymbol{V}_1 + \boldsymbol{V}_2) &= \tfrac{1}{2}(\boldsymbol{V}_1' + \boldsymbol{V}_2') = \frac{1}{\sqrt{2}} \boldsymbol{V}; \\ \boldsymbol{V}_1 - \boldsymbol{V}_2 &= \sqrt{2}\gamma \boldsymbol{e}, \quad \boldsymbol{V}_1' - \boldsymbol{V}_2' = \sqrt{2}\gamma \boldsymbol{e}'. \end{aligned} \tag{1.8}$$

Due to (1.7), one has

$$f_{01} f_{02} \mathrm{d}^3 V_1 \mathrm{d}^3 V_2 = n_0^2 \pi^{-3} e^{-V^2 - \gamma^2} \mathrm{d}^3 V \gamma^2 \mathrm{d}\gamma \, \mathrm{d}^2 e, \tag{1.9}$$

where $\mathrm{d}^2 e$ is the solid angle element of relative velocity.

The collision term of the transport equation for particles with spin contains the scattering amplitude and its adjoint separately rather than their combination as a scattering cross section. The scattering amplitude a depends on the relative kinetic energy, the directions of the relative velocities before and after the collision (i.e. on $\boldsymbol{e}'$ and $\boldsymbol{e}$) and on the spins $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ of both colliding particles:

$$a = a(k T_0 \gamma^2; \; \boldsymbol{e}', \boldsymbol{e}; \; \boldsymbol{s}_1, \boldsymbol{s}_2) . \tag{1.10}$$

In the following, we shall not indicate the dependence on the energy and the spins explicitly but simply write $a(\boldsymbol{e}', \boldsymbol{e})$ instead of (1.10).

Now we are prepared to write down the collision bracket $\langle \Psi \omega(\Phi) \rangle_0$ for two functions Ψ and Φ in a form which, although different in appearance, is equivalent to that used in I. With the help of the optical theorem, the collision bracket will be divided into two parts. The first one is bilinear in the scattering amplitude and its adjoint and yields the classical Boltzmann collision term for spinless particles; the second term contains the "real part of the scattering amplitude"

$$\operatorname{Re} a = \tfrac{1}{2}(a + a^\dagger) \tag{1.11}$$

in the forward direction and vanishes for spinless particles. Introducing the thermal velocity

$$c_0 = \sqrt{\frac{k T_0}{2 \pi m_{12}}}, \quad m_{12} = \tfrac{1}{2} m , \tag{1.12}$$

and the abbreviations

$$\Phi_i = \Phi(\boldsymbol{V}_i, \boldsymbol{s}_i), \quad \Phi_i' = \Phi(\boldsymbol{V}_i', \boldsymbol{s}_i), \quad (i = 1, 2) \tag{1.13}$$

one has

$$\begin{aligned} \langle \Psi \omega(\Phi) \rangle_0 = n_0 c_0 \frac{8}{\pi^{3/2}} \int \mathrm{d}^3 V e^{-V^2} \{ & \Psi_1 \tfrac{1}{2} a(\boldsymbol{e}, \boldsymbol{e}') [a^\dagger(\boldsymbol{e}', \boldsymbol{e})(\Phi_1 + \Phi_2) - (\Phi_1' + \Phi_2') a^\dagger(\boldsymbol{e}', \boldsymbol{e})] \\ & + \Psi_1 \tfrac{1}{2} [(\Phi_1 + \Phi_2) a(\boldsymbol{e}, \boldsymbol{e}') - a(\boldsymbol{e}, \boldsymbol{e}')(\Phi_1' + \Phi_2')] a^\dagger(\boldsymbol{e}', \boldsymbol{e}) \} + \langle \Psi \tilde{\omega}(\Phi) \rangle_0 . \end{aligned} \tag{1.14}$$

Here the curly bracket $\{\dots\}$ denotes the following average:

$$\{\dots\} = \int_0^\infty \mathrm{d}\gamma \, \gamma^3 e^{-\gamma^2} \frac{1}{4\pi} \iint \mathrm{d}^2 e \, \mathrm{d}^2 e' \frac{\operatorname{tr}_1 \operatorname{tr}_2}{(2S+1)^2} \cdots . \tag{1.15}$$

The part of the collision bracket linear in the real part of the scattering amplitude is

$$\langle \Psi \tilde{\omega}(\Phi) \rangle_0 = n_0 \pi^{-3} \iint \mathrm{d}^3 V \, \mathrm{d}\gamma \, \gamma^2 e^{-V^2 - \gamma^2} \int \mathrm{d}^2 e \frac{\operatorname{tr}_1 \operatorname{tr}_2}{(2S+1)^2} \frac{h}{2 m_{12}} \operatorname{Re} a(\boldsymbol{e}, \boldsymbol{e}) \frac{1}{i} [(\Phi_1 + \Phi_2), (\Psi_1 + \Psi_2)] . \tag{1.16}$$

The bracket $[A, B]$ denotes the commutator AB − BA. This term gives no contribution to the collision bracket if (at least) one of the functions Ψ_1, Φ_1 involved does not depend on the spin or if Ψ_1 equals Φ_1 apart from a spin-independent factor.

For the case where Φ does not depend on the spin one has

$$\langle \Psi\,\omega(\Phi)\rangle_0 = n_0\,c_0\,\frac{8}{\pi^{3/2}}\int \mathrm{d}^3V\,e^{-V^2}\{\Psi_1\,\sigma(\boldsymbol{e},\boldsymbol{e}')\,\delta(\Phi)\}, \tag{1.17}$$

with

$$\delta(\Phi) = \Phi_1 + \Phi_2 - \Phi_1' - \Phi_2'. \tag{1.18}$$

The "cross section"

$$\sigma(\boldsymbol{e},\boldsymbol{e}') = a(\boldsymbol{e},\boldsymbol{e}')\,a^{\dagger}(\boldsymbol{e}',\boldsymbol{e}) \tag{1.19}$$

still depends on the spins $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$. Evaluating a collision bracket $\langle \Psi\,\omega(\Phi)\rangle_0$ according to (1.14) or (1.17) one can immediately integrate over the center of mass velocity $\boldsymbol{V}$ since the scattering amplitude does not depend on it (due to Galilean invariance). As we shall see in § 3, 4, some conclusions concerning the magnitude of various relaxation coefficients responsible for the coupling of velocity-spin-correlation moments with those underlying the heat flux and the friction pressure can already be drawn at that stage.

§ 2. Relaxation Coefficients for Mean Values of Spin-Independent Tensors

The heat flux q_μ and the friction pressure $p_{\mu\nu}$ are proportional to the expansion coefficients $a_\mu^{(3)}$ and $a_{\mu\nu}^{(1)}$ defined in I. These in turn are the mean values of the expansion tensors

$$\Phi_{1\mu}^{(3)} = \frac{2}{\sqrt{5}}\left(V_1^2 - \frac{5}{2}\right)V_{1\mu} \tag{2.1}$$

and

$$\Phi_{1\mu\nu}^{(1)} = \sqrt{2}\;\overline{V_{1\mu}V_{1\nu}}\,, \tag{2.2}$$

where $\overline{a_\mu b_\nu}$ denotes the second rank irreducible part of the tensor $a_\mu b_\nu$:

$$\overline{a_\mu b_\nu} = \tfrac{1}{2}(a_\mu b_\nu + a_\nu b_\mu) - \tfrac{1}{3}\,\boldsymbol{a}\cdot\boldsymbol{b}\,\delta_{\mu\nu}. \tag{2.3}$$

Without the small corrections (of order of the thermal diffusion factor) which arise from higher Sonine polynomials, the diagonal relaxation coefficients $\omega_{-1}^{(33)}$ and $\omega_{+2}^{(11)}$ would determine the heat conductivity and the viscosity, respectively, of a gas consisting of particles with spin if no correlations between velocity and spin would be set up by the irreversible processes. The nondiagonal relaxation coefficients $\omega_{-1}^{(k3)}$ and $\omega_{+2}^{(k1)}$, $k \neq 3, 1$ respectively, which lead to such correlations (by coupling mean values of spin-independent tensors to mean values of spin-dependent tensors) shall be studied in the following sections.

Here we shall show that the collision brackets for spin-independent tensors can be related to certain integrals $\overline{\Omega}^{(l,r)}$ which are analogous to Chapman's integrals $\Omega^{(l,r)}$ of the monatomic gas[11] but with the scattering cross section of spinless particles replaced by a spin-averaged cross section $\bar{\sigma}$.

For the evaluation of $\omega_{-1}^{(33)}$ and $\omega_{+2}^{(11)}$ we use the collision bracket in the form (1.17). Due to (1.7) and (2.1, 2) we have

$$\delta(\Phi_\mu^{(3)}) = 2\sqrt{\frac{2}{5}}\,\gamma^2(e_\mu e_\varrho - e_\mu' e_\varrho')\,V_\varrho\,, \tag{2.4}$$

and

$$\delta(\Phi_{\mu\nu}^{(1)}) = \sqrt{2}\,\gamma^2(\overline{e_\mu e_\nu} - \overline{e_\mu' e_\nu'})\,. \tag{2.5}$$

Carrying out the integration over $\boldsymbol{V}$ we find

$$\begin{aligned}\omega_{-1}^{(33)} &= \frac{1}{3}\langle \Phi_\mu^{(3)}\,\omega(\Phi_\mu^{(3)})\rangle_0 \\ &= \frac{16}{15}\,n_0\,c_0\{\gamma^4\,\sigma(\boldsymbol{e},\boldsymbol{e}')\,(1-(\boldsymbol{e}\cdot\boldsymbol{e}')^2)\}\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\omega_{+2}^{(11)} &= \frac{1}{5}\langle \Phi_{\mu\nu}^{(1)}\,\omega(\Phi_{\mu\nu}^{(1)})\rangle_0 \\ &= \frac{8}{5}\,n_0\,c_0\{\gamma^4\,\sigma(\boldsymbol{e},\boldsymbol{e}')\,(1-(\boldsymbol{e}\cdot\boldsymbol{e}')^2)\}\,,\end{aligned} \tag{2.7}$$

and this clearly shows that

$$\omega_{+2}^{(11)} = \frac{3}{2}\,\omega_{-1}^{(33)}\,. \tag{2.8}$$

Now we define

$$\begin{aligned}&\overline{\Omega}^{(l,r)} \\ &= c_0\int_0^\infty \mathrm{d}\gamma\,\gamma^{3+2r}\,e^{-\gamma^2}\cdot 2\pi\int_0^\pi \sin\chi\,\mathrm{d}\chi\,\mathrm{d}\chi\,\bar{\sigma}\,(1-\cos^l\chi)\,,\end{aligned} \tag{2.9}$$

where χ is the angle between $\boldsymbol{e}'$ and $\boldsymbol{e}$:

$$\cos\chi = \boldsymbol{e}\cdot\boldsymbol{e}'\,, \tag{2.10}$$

and $\bar{\sigma} = \bar{\sigma}(k\,T_0\,\gamma^2,\,\chi)$ is the spin-averaged differential cross section

$$\bar{\sigma} = \frac{\mathrm{tr}_1\,\mathrm{tr}_2}{(2S+1)^2}\,\sigma(\boldsymbol{e},\boldsymbol{e}')\,. \tag{2.11}$$

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[11] S. Chapman and T. G. Cowling, The Mathematical Theory of Non-Uniform Gases, Cambridge 1939. — J. O. Hirschfelder, C. F. Curtiss, and R. B. Bird, Molecular Theory of Gases and Liquids, Wiley, New York 1954. L. Waldmann, Transporterscheinungen in Gasen von mittlerem Druck, in Handbuch der Physik, Vol. 12, ed. S. Flügge, Springer, Berlin 1958.

To find the connection between these $\bar{\Omega}$-integrals and curly bracket integrals like that occurring in (2.6), we note that $(4\pi)^{-1} \int\int \mathrm{d}^2 e \, \mathrm{d}^2 e' \ldots$ can be replaced by $2\pi \int_0^\pi \sin\chi \, \mathrm{d}\chi \ldots$ if the function to be integrated depends on $\boldsymbol{e}$ and $\boldsymbol{e}'$ via $\boldsymbol{e} \cdot \boldsymbol{e}'$ only. Recalling the definition (1.15) we find the identity

$$c_0 \{\gamma^{2r} \sigma(\boldsymbol{e}, \boldsymbol{e}') (1 - (\boldsymbol{e} \cdot \boldsymbol{e}')^l)\} = \bar{\Omega}^{(l,r)}. \tag{2.12}$$

Thus we have

$$\omega_{+2}^{(11)} = \frac{3}{2} \omega_{-1}^{(33)} = \frac{8}{5} n_0 \bar{\Omega}^{(2,2)}. \tag{2.13}$$

The scattering cross section $\sigma(\boldsymbol{e}, \boldsymbol{e}')$ can be divided into a spin-independent part $\sigma_0(\boldsymbol{e} \cdot \boldsymbol{e}')$ and a spin dependent part. Later on we shall need integrals like (2.9) where the spin-dependent part of $\sigma(\boldsymbol{e}, \boldsymbol{e}')$ has been neglected, i.e. where σ_0 replaces $\bar{\sigma}$. For these we shall write $\Omega^{(l,r)}$ (without the bar).

§ 3. Relaxation Coefficients Coupling the Heat Flux to Mean Values of Spin-Dependent Vectors

In this section we are concerned with the relaxation coefficients

$$\omega_{-1}^{(k3)} = \frac{1}{3} \langle \Phi_\mu^{(k)} \, \omega(\Phi_\mu^{(3)}) \rangle_0 \tag{3.1}$$

$(k \neq 3)$ which, in the TRE, couple the heat flux with mean values of the following spin-dependent expansion vectors:

$$\begin{aligned}
\Phi_\mu^{(2)} &= \frac{\sqrt{3}}{S_0} (\boldsymbol{V} \times \boldsymbol{s})_\mu , \\
\Phi_\mu^{(4)} &= \frac{3}{S_0 S_1} \overline{s_\mu s_\nu} V_\nu , \\
\Phi_\mu^{(5)} &= \sqrt{\frac{6}{5}} \frac{1}{S_0} \left(V^2 - \frac{5}{2} \right) (\boldsymbol{V} \times \boldsymbol{s})_\mu , \\
\Phi_\mu^{(6)} &= 3 \sqrt{\frac{2}{5}} \frac{1}{S_0 S_1} \left(V^2 - \frac{5}{2} \right) \overline{s_\mu s_\nu} V_\nu , \\
\Phi_\mu^{(7)} &= \sqrt{\frac{30}{7}} \frac{1}{S_0 S_1} \overline{V_\mu V_\nu V_\lambda} s_\nu s_\lambda ,
\end{aligned} \tag{3.2}$$

with

$$\overline{V_\mu V_\nu V_\lambda} = V_\mu V_\nu V_\lambda - \tfrac{1}{5} V^2 (V_\mu \delta_{\nu\lambda} + V_\nu \delta_{\mu\lambda} + V_\lambda \delta_{\mu\nu}) .$$

The abbreviations

$$S_0 = \sqrt{S(S+1)}, \quad S_1 = \sqrt{S(S+1) - \tfrac{3}{4}} \tag{3.3}$$

have been used. The vectors $\boldsymbol{\Phi}^{(2)}$ and $\boldsymbol{\Phi}^{(5)}$ are odd in $\boldsymbol{s}$ and even under time reversal ($\boldsymbol{V}, \boldsymbol{s} \to -\boldsymbol{V}, -\boldsymbol{s}$). The other vectors $\boldsymbol{\Phi}^{(4)}$, $\boldsymbol{\Phi}^{(6)}$, and $\boldsymbol{\Phi}^{(7)}$ contain the second rank tensor $\overline{\boldsymbol{s}\boldsymbol{s}}$ and are odd under time reversal. The last two vectors of (3.2) did not occur in I. The time reversal invariance of the scattering amplitude implies Onsager-Casimir relations for the nondiagonal relaxation coefficients (see I). In particular we have

$$\omega_{-1}^{(k3)} = -\omega_{-1}^{(3k)} \quad \text{for} \quad k = 2, 5 , \tag{3.4}$$

and

$$\omega_{-1}^{(k3)} = \omega_{-1}^{(3k)} \quad \text{for} \quad k = 4, 6, 7 . \tag{3.5}$$

Firstly, we want so show, that $\omega_{-1}^{(23)}$ vanishes for a pure gas of spin particles. To this end we write down $\omega_{-1}^{(23)}$ explicitly, using (1.7), (1.17) and (2.4):

$$\omega_{-1}^{(23)} = \frac{2}{3} \sqrt{\frac{3}{5}} \frac{1}{S_0} n_0 \frac{8}{\pi^{3/2}} c_0 \int \mathrm{d}^3 V e^{-V^2} \cdot \{\varepsilon_{\mu\nu\lambda} (V_\nu + \gamma e_\nu) s_\lambda \sigma(\boldsymbol{e}, \boldsymbol{e}') \gamma^2 (e_\mu e_\varrho - e'_\mu e'_\varrho) V_\varrho\} .$$

Noting that

$$\int \mathrm{d}^3 V e^{-V^2} V_\varrho = 0; \qquad \pi^{-3/2} \int \mathrm{d}^3 V e^{-V^2} V_\nu V_\varrho = \tfrac{1}{2} \delta_{\nu\varrho}$$

and $\varepsilon_{\mu\nu\lambda} e_\mu e_\nu = \varepsilon_{\mu\nu\lambda} e'_\mu e'_\nu = 0$, one immediately finds

$$\omega_{-1}^{(23)} = -\omega_{-1}^{(32)} = 0 . \tag{3.6}$$

Thus we arrive at the conclusion that there is no (direct) coupling between heat flux and "azimuthal spin" $\langle \boldsymbol{V} \times \boldsymbol{s} \rangle$ in a pure gas of particles with spin. No assumptions concerning the spin dependence of the scattering amplitude have been made. This result is typical for a pure gas. With a mixture of spin particles, the azimuthal spin may be coupled to both the heat and diffusion flows.

Next we want to express the other $\omega_{-1}^{(k3)}$ ($k = 4 - 7$) by curly bracket integrals (which are non-zero in general).

Integrating over $\boldsymbol{V}$ one finds:

$$\begin{aligned}
\omega_{-1}^{(43)} &= \omega_{-1}^{(34)} \\
&= \frac{8}{\sqrt{5}} \frac{n_0 c_0}{S_0 S_1} \{\gamma^2 \overline{s_{1\mu} s_{1\nu}} \sigma(\boldsymbol{e}, \boldsymbol{e}') (e_\mu e_\nu - e'_\mu e'_\nu)\} ,
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
\omega_{-1}^{(53)} &= -\omega_{-1}^{(35)} \\
&= -\frac{4}{5} \sqrt{\frac{2}{3}} \frac{n_0 c_0}{S_0} \{\gamma^4 s_{1\lambda} \sigma(\boldsymbol{e}, \boldsymbol{e}') \varepsilon_{\lambda\mu\nu} e'_\mu e_\nu \boldsymbol{e}' \cdot \boldsymbol{e}\}.
\end{aligned} \tag{3.8}$$

Dividing $\sigma(\boldsymbol{e}, \boldsymbol{e}')$ into a part

$$\sigma_s(\boldsymbol{e}, \boldsymbol{e}') = \sigma_s(\boldsymbol{e}', \boldsymbol{e}) = \tfrac{1}{2} (\sigma(\boldsymbol{e}, \boldsymbol{e}') + \sigma(\boldsymbol{e}', \boldsymbol{e})) \tag{3.9}$$

which is symmetric under the interchange of $\boldsymbol{e}$ and $\boldsymbol{e}'$ and an anti-symmetric part

$$\sigma_a(\boldsymbol{e}, \boldsymbol{e}') = -\sigma_a(\boldsymbol{e}', \boldsymbol{e}) = \tfrac{1}{2} (\sigma(\boldsymbol{e}, \boldsymbol{e}') - \sigma(\boldsymbol{e}' \, \boldsymbol{e})) , \tag{3.10}$$

one sees that in both of the integrals (3.7) and (3.8) σ_a only gives a contribution. Thus we may rewrite

$$\omega_{-1}^{(43)} = \frac{16}{\sqrt{5}} \frac{n_0 c_0}{S_0 S_1} \{\gamma^2 \overline{s_{1\mu} s_{1\nu}} e_\mu e_\nu \sigma_a(\boldsymbol{e}, \boldsymbol{e}')\}, \tag{3.11}$$

and

$$\omega_{-1}^{(53)} = -\frac{4}{5} \sqrt{\frac{2}{3}} \frac{n_0 c_0}{S_0} \{\gamma^4 \cos\chi \sin\chi \, \boldsymbol{n} \cdot \boldsymbol{s}_1 \sigma_a(\boldsymbol{e}, \boldsymbol{e}')\}, \tag{3.12}$$

where the unit vector

$$\boldsymbol{n} = (\sin\chi)^{-1} \boldsymbol{e}' \times \boldsymbol{e}, \tag{3.13}$$

which is perpendicular to the collision plane, has been introduced.

As we shall see later, there are strong indications that for collisions of nonpolar molecules σ_a is smaller than σ_s by at least a factor 10. Therefore we may neglect σ_a in collision brackets, where σ_s already gives a non-vanishing contribution. This is the case with $\omega_{-1}^{(63)}$ and $\omega_{-1}^{(73)}$ for which one finds

$$\omega_{-1}^{(63)} = \omega_{-1}^{(36)} = \frac{8}{5} \sqrt{2} \frac{n_0 c_0}{S_0 S_1} \cdot \{\gamma^4 e_\mu \overline{s_{1\mu} s_{1\nu}} (e_\nu - e'_\nu \cos\chi) \sigma_s(\boldsymbol{e}, \boldsymbol{e}')\} + O(\sigma_a), \tag{3.14}$$

and

$$\omega_{-1}^{(73)} = \omega_{-1}^{(37)} = 4 \sqrt{\frac{6}{7}} \frac{n_0 c_0}{S_0 S_1} \cdot \{\gamma^4 \overline{s_{1\mu} s_{1\nu}} (\overline{e_\mu e_\nu e_\lambda} e_\lambda - \overline{e_\mu e_\nu e'_\lambda} e'_\lambda) \sigma_s(\boldsymbol{e}, \boldsymbol{e}')\} + O(\sigma_a). \tag{3.15}$$

Here $O(\sigma_a)$ stands for terms which contain σ_a but not σ_s. Since

$$\overline{s_\mu s_\nu} (\overline{e_\mu e_\nu e_\lambda} e_\lambda - \overline{e_\mu e_\nu e'_\lambda} e'_\lambda) = \frac{2}{5} \overline{s_\mu s_\nu} (e_\mu e_\nu - \cos\chi \, e'_\mu e_\nu) - \frac{2}{15} \overline{s_\mu s_\nu} (e_\mu e_\nu - e'_\mu e'_\nu) \tag{3.16}$$

we have, in this approximation,

$$\omega_{-1}^{(73)} = \sqrt{\frac{3}{7}} \, \omega_{-1}^{(63)} + O(\sigma_a). \tag{3.17}$$

Thus we see that both $\boldsymbol{\Phi}^{(6)}$ and $\boldsymbol{\Phi}^{(7)}$ will be of equal importance for the Senftleben-Beenakker effect.

§ 4. Relaxation Coefficients Coupling the Friction Pressure to Mean Values of Spin-Dependent Tensors

Now we shall deal with the "tensor" relaxation coefficients

$$\omega_{+2}^{(k1)} = \frac{1}{5} \langle \Phi_{\mu\nu}^{(k)} \, \omega(\Phi_{\mu\nu}^{(1)}) \rangle_0, \tag{4.1}$$

$(k \neq 1)$ which, in the TRE, couple the friction pressure tensor with mean values of the following spin-dependent expansion tensors:

$$\begin{aligned} \Phi_{\mu\nu}^{(2)} &= \sqrt{\frac{15}{2}} \frac{1}{S_0 S_1} \overline{s_\mu s_\nu}, \\ \Phi_{\mu\nu}^{(3)} &= \frac{2}{S_0} \overline{\varepsilon_{\mu\lambda\varkappa} \overline{V_\nu V_\lambda} s_\varkappa}, \\ \Phi_{\mu\nu}^{(5)} &= \frac{\sqrt{5}}{S_0 S_1} \left(V^2 - \frac{3}{2}\right) \overline{s_\mu s_\nu}, \\ \Phi_{\mu\nu}^{(6)} &= 6 \sqrt{\frac{5}{7}} \frac{1}{S_0 S_1} \overline{\overline{V_\mu V_\varrho} \, \overline{s_\varrho s_\nu}}, \\ \Phi_{\mu\nu}^{(7)} &= \frac{5}{3} \frac{1}{S_0 S_1} \overline{V_\mu V_\nu V_\lambda V_\varkappa} s_\lambda s_\varkappa. \end{aligned} \tag{4.2}$$

The tensor $\Phi_{\mu\nu}^{(3)}$ is odd in $\boldsymbol{s}$ and its time reversal behavior is odd, opposite to that of $\Phi_{\mu\nu}^{(1)}$; the other four tensors contain the spin tensor $\overline{s_\mu s_\nu}$ and are even under time reversal. As always, the bar $\overline{}$ denotes the (traceless) irreducible part of a tensor. Here we have the following Onsager-Casimir relations

$$\omega_{+2}^{(k1)} = \omega_{+2}^{(1k)} \quad \text{for} \quad k = 2, 5, 6, 7, \tag{4.3}$$

$$\omega_{+2}^{(31)} = -\omega_{+2}^{(13)}. \tag{4.4}$$

The coupling between the tensor polarization $\langle \overline{s_\mu s_\nu} \rangle$ and the friction pressure is determined by $\omega_{+2}^{(21)}$. Using (1.17) and (2.5) one immediately obtains

$$\begin{aligned} \omega_{+2}^{(21)} &= \omega_{+2}^{(12)} \\ &= 8 \sqrt{\frac{3}{5}} \frac{n_0 c_0}{S_0 S_1} \{\gamma^2 \overline{s_{1\mu} s_{1\nu}} \sigma(\boldsymbol{e}, \boldsymbol{e}') (e_\mu e_\nu - e'_\mu e'_\nu)\} \\ &= 16 \sqrt{\frac{3}{5}} \frac{n_0 c_0}{S_0 S_1} \{\gamma^2 \overline{s_{1\mu} s_{1\nu}} e_\mu e_\nu \sigma_a(\boldsymbol{e}, \boldsymbol{e}')\}. \end{aligned} \tag{4.5}$$

Clearly this relaxation coefficients can be expected to be rather small since σ_a only gives a contribution in (4.5). Comparing (4.5) with (3.11) we have exactly:

$$\omega_{+2}^{(21)} = \sqrt{3} \, \omega_{-1}^{(43)}. \tag{4.6}$$

Hence we arrive at the important conclusion that the coupling between tensor polarization and friction pressure is of equal order as that between the longitudinal component of the tensor polarization $\langle \overline{\boldsymbol{s}\boldsymbol{s}} \cdot \boldsymbol{V} \rangle$ (Kagan vector) and the (translational) heat flux. Similarly one finds

$$\begin{aligned} \omega_{+2}^{(51)} &= \omega_{+2}^{(15)} \\ &= 8 \sqrt{\frac{2}{5}} \frac{n_0 c_0}{S_0 S_1} \{\gamma^2 (\gamma^2 - \tfrac{3}{2}) \overline{s_{1\mu} s_{1\nu}} e_\mu e_\nu \sigma_a(\boldsymbol{e}, \boldsymbol{e}')\}. \end{aligned} \tag{4.7}$$

Again σ_a only gives a contribution to this relaxation coefficient. For a gas-mixture, however, one obtains a relaxation coefficient containing σ_s which couples the "weighted" tensor polarization $\langle (V^2 - \frac{3}{2})\, \overline{s_\mu s_\nu} \rangle$ to the friction pressure.

Next, we note that $\omega_{+2}^{(31)}$ is proportional to $\omega_{-1}^{(53)}$ much as $\omega_{+2}^{(21)}$ equals $\omega_{-1}^{(43)}$ apart from a constant factor. A short calculation yields

$$\omega_{+2}^{(31)} = -\,\omega_{+2}^{(13)} = -\frac{8}{5}\sqrt{2}\,\frac{n_0 c_0}{S_0} \cdot \{\gamma^4 s_{1\lambda}\, \sigma_a(\boldsymbol{e}, \boldsymbol{e}')\, \varepsilon_{\lambda\mu\nu}\, e_\mu e_\nu\, \boldsymbol{e}' \cdot \boldsymbol{e}\}\,. \tag{4.8}$$

Comparing this with (3.8) we have exactly

$$\omega_{+2}^{(31)} = -\,\omega_{+2}^{(13)} = 2\sqrt{3}\,\omega_{-1}^{(53)}\,. \tag{4.9}$$

Finally we shall show that $\omega_{+2}^{(61)}$ and $\omega_{+2}^{(71)}$ can be expressed by collision brackets containing σ_s. Integrating over $\boldsymbol{V}$, one obtains

$$\omega_{+2}^{(61)} = \omega_{+2}^{(16)} = \frac{24}{5}\sqrt{\frac{10}{7}}\,\frac{n_0 c_0}{S_0 S_1} \cdot \{\gamma^4\, \overline{e_\varrho e_\mu}\; \overline{s_{1\mu} s_{1\nu}}\, \sigma(\boldsymbol{e}, \boldsymbol{e}')\, (\overline{e_\varrho e_\nu} - \overline{e_\varrho' e_\nu'})\}\,. \tag{4.10}$$

Since

$$\overline{s_\mu s_\nu}\; \overline{e_\mu e_\varrho}\, (\overline{e_\varrho e_\nu} - \overline{e_\varrho' e_\nu'}) = \overline{s_\mu s_\nu}\, (e_\mu e_\nu - \boldsymbol{e}' \cdot \boldsymbol{e}\, e_\mu e_\nu') - \tfrac{1}{3}\, \overline{s_\mu s_\nu}\, (e_\mu e_\nu - e_\mu' e_\nu') \tag{4.11}$$

we find

$$\omega_{+2}^{(61)} = \frac{24}{5}\sqrt{\frac{10}{7}}\,\frac{n_0 c_0}{S_0 S_1} \cdot \{\gamma^4\, \overline{s_{1\mu} s_{1\nu}}\, (e_\mu e_\nu - \boldsymbol{e}' \cdot \boldsymbol{e}\, e_\mu e_\nu')\, \sigma_s(\boldsymbol{e}, \boldsymbol{e}')\} + O(\sigma_a)\,. \tag{4.12}$$

Comparison with (3.14) yields the approximate relation

$$\omega_{+2}^{(61)} = 3\sqrt{\frac{5}{7}}\,\omega_{-1}^{(63)}\,. \tag{4.13}$$

Similarly to (4.10), one obtains

$$\omega_{+2}^{(71)} = \frac{2}{3}\sqrt{2}\,\frac{n_0 c_0}{S_0 S_1} \cdot \{\gamma^6\, \overline{e_\lambda e_\varkappa e_\mu e_\nu}\; \overline{s_{1\mu} s_{1\nu}}\, \sigma(\boldsymbol{e}, \boldsymbol{e}')\, (e_\lambda e_\varkappa - e_\lambda' e_\varkappa')\}\,. \tag{4.14}$$

Due to the relations

$$\overline{e_\mu e_\nu e_\lambda e_\varkappa}\, e_\lambda e_\varkappa = \frac{12}{35}\, \overline{e_\mu e_\nu}$$

and

$$\overline{e_\mu e_\nu e_\lambda e_\varkappa}\, e_\lambda' e_\varkappa' = \left((\boldsymbol{e} \cdot \boldsymbol{e}')^2 - \frac{1}{7}\right) \overline{e_\mu e_\nu} - \frac{4}{7}\, \boldsymbol{e} \cdot \boldsymbol{e}'\, \overline{e_\mu e_\nu'} + \frac{2}{35}\, \overline{e_\mu' e_\nu'}\,, \tag{4.15}$$

we find

$$\omega_{+2}^{(71)} = \frac{2\sqrt{2}}{3}\,\frac{n_0 c_0}{S_0 S_1}\, \{\gamma^6\, \overline{s_{1\mu} s_{1\nu}}\, [e_\mu e_\nu (1 - (\boldsymbol{e} \cdot \boldsymbol{e}')^2) - \tfrac{4}{7}\, (e_\mu e_\nu - \boldsymbol{e} \cdot \boldsymbol{e}'\, e_\mu e_\nu')]\, \sigma_s(\boldsymbol{e}, \boldsymbol{e}')\} + O(\sigma_a)\,. \tag{4.16}$$

§ 5. Scattering Amplitude, Order of Magnitude of the Coupling Relaxation Coefficients

Now we want to determine the relative importance of the various nondiagonal relaxation coefficients $\omega_{-1}^{(k3)}$ and $\omega_{+2}^{(k1)}$ which have been expressed by curly bracket integrals in the preceding sections. All these relaxation coefficients leading to a correlation between velocity and spin are zero unless the scattering amplitude contains a "nonspherical" part. Not all of them are of equal importance for the Senftleben effect.

A two-particle operator like the scattering amplitude or the two-particle interaction potential is called "spherical" or "nonspherical", respectively, if it does or does not commute with the total spin $\boldsymbol{s}_1 + \boldsymbol{s}_2$.

As a measure of the nonsphericity we introduce the scaling factor ε which determines the order of magnitude of the nonspherical part of the two-particle interaction potential as compared with its spherical part. The nonspherical part of the scattering amplitude then contains terms of order ε and higher powers in ε. As a typical value, one may expect $\varepsilon \approx 0.1$. The interaction potential for two ortho-H_2 molecules at low temperature (i.e. spin 1) contains two nonspherical terms which are proportional to [10]

$$\varepsilon\, x_\mu x_\nu S_{\mu\nu} \tag{5.1}$$

with

$$S_{\mu\nu} = S_1^{-2}\, (\overline{s_{1\mu} s_{1\nu}} + \overline{s_{2\mu} s_{2\nu}})\,, \tag{5.2}$$

and

$$\varepsilon\, x_\mu x_\nu x_\lambda x_\varkappa\, \overline{s_{1\mu} s_{1\nu}}\; \overline{s_{2\lambda} s_{2\varkappa}}\,. \tag{5.3}$$

Here $\boldsymbol{x}$ is the vector pointing from the c.m. of molecule 1 to the c.m. of molecule 2. It is presumed that the interaction does not depend on velocities (no spin-orbit coupling). This is always true for slow particles. A term of the type (5.1) can be expected to be a common feature in the interaction of rotating diatomic molecules, although with higher spins additional terms with higher rank spin — tensors may occur.

Then the scattering amplitude has the form [12]

$$a = a_0 + \varepsilon\, A_{\mu\nu} S_{\mu\nu} + \varepsilon\, B_{\mu\nu,\lambda\varkappa}\, \overline{s_{1\mu} s_{1\nu}}\; \overline{s_{2\lambda} s_{2\varkappa}} + O(\varepsilon^2)\,. \tag{5.4}$$

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[12] For a detailed discussion of the nonspherical potential and the spin dependence of the scattering amplitude of rotating diatomic molecules, see Ref. [10].

Here a_0 is the spherical part of the scattering amplitude. The tensors $A_{..}$ and $B_{..}$ are proportional to Cartesian tensors constructed from the components of $\boldsymbol{e}$ and $\boldsymbol{e}'$ which are symmetric under the exchange $\boldsymbol{e}, \boldsymbol{e}' \rightarrow \boldsymbol{e}', \boldsymbol{e}$. In particular, we have

$$A_{\mu\nu} = a_1 (\overline{e_\mu e_\nu'} + \overline{e_\mu' e_\nu}) + a_2 (\overline{e_\mu e_\nu} + \overline{e_\mu' e_\nu'}) \,. \quad (5.5)$$

The scalar coefficients a_1 and a_2 — just as a_0 — depend on the relative kinetic energy (i.e. on γ^2) and the angle of deflection χ. An explicit expression for the tensor $B_{...}$ corresponding to (5.5) is not needed since the B-term gives no contribution of order ε to the relaxation coefficients in which we are interested.

Terms of the scattering amplitude which are antisymmetric under the interchange $\boldsymbol{e}, \boldsymbol{e}' \rightarrow \boldsymbol{e}', \boldsymbol{e}$, e.g., a term proportional to

$$\boldsymbol{n} \cdot (\boldsymbol{s}_1 + \boldsymbol{s}_2) \,,$$

are of order ε^2. Indeed, such terms appear only in the second Born approximation. This holds for any parity and time reversal invariant interaction potential which does not depend on the relative momentum of the colliding particles; in particular, it is true for potentials of the type (5.1) and (5.3).

Hence nondiagonal collision brackets involving σ_s can be expected to be of order ε whereas those containing σ_a will be at least of order ε^2.

By using the scattering amplitude (5.4) and evaluating the spin traces (cf. I) one indeed finds:

$$\omega_{-1}^{(43)} = \frac{1}{\sqrt{3}} \omega_{+2}^{(21)} = O(\varepsilon^3); \quad \omega_{+2}^{(51)} = O(\varepsilon^3);$$

$$\omega_{-1}^{(53)} = \frac{1}{2\sqrt{3}} \omega_{+2}^{(31)} = O(\varepsilon^2); \quad (5.6)$$

$$\begin{aligned} \omega_{-1}^{(63)} &= \sqrt{\frac{7}{3}}\, \omega_{-1}^{(73)} = \frac{1}{3} \sqrt{\frac{7}{5}}\, \omega_{+2}^{(61)} \\ &= \varepsilon \frac{16}{75} \sqrt{2}\, S_0 S_1 n_0 c_0 \{\gamma^4 (e_\mu e_\nu - \boldsymbol{e} \cdot \boldsymbol{e}' e_\mu e_\nu') \\ &\quad \cdot (a_0 A_{\mu\nu}^* + a_0^* A_{\mu\nu})\} + O(\varepsilon^2) \,, \end{aligned} \quad (5.7)$$

$$\begin{aligned} \omega_{+2}^{(71)} = \varepsilon \frac{4\sqrt{2}}{45} S_0 S_1 n_0 c_0 \{\gamma^6 [e_\mu e_\nu (1 - \cos^2 \chi) \\ - \tfrac{4}{7} (e_\mu e_\nu - \cos \chi\, e_\mu e_\nu')] (a_0 A_{\mu\nu}^* + a_0^* A_{\mu\nu})\} + O(\varepsilon^2) \,. \end{aligned} \quad (5.8)$$

In (5.7, 8) the curly bracket (see 1.15) can be replaced by

$$\{\ldots\} = \int d\gamma\, \gamma^3 e^{-\gamma^2} \frac{1}{4\pi} \int \int d^2e\, d^2e' \ldots$$

due to

$$\mathrm{tr}_1\, \mathrm{tr}_2 = 1\, (2S + 1)^2 \,.$$

Note that, in order ε, only the A-term of the scattering amplitude resulting from an interaction potential of type (5.1), but not the B-term, resulting from a potential of type (5.3), contributes to the relaxation coefficients (5.7) and (5.8). The reason is that the B-term of (5.4) contains both of the spins as factors so that up to terms of order ε vanishing traces "tr_2" occur.

Using (5.5) one may rewrite (5.7) and (5.8)

$$\omega_{-1}^{(63)} = \varepsilon \frac{32}{225} \sqrt{2}\, S_0 S_1 n_0 (\Omega_{01}^{(2,2)} + \Omega_{02}^{(2,2)}) \,, \quad (5.9)$$

$$\omega_{+2}^{(71)} = \varepsilon \frac{8\sqrt{2}}{45} S_0 S_1 n_0 \left(\frac{8}{7} \Omega_{01}^{(2,3)} + \Omega_{02}^{(4,3)} - \frac{6}{7} \Omega_{02}^{(2,3)} \right) \quad (5.10)$$

where Ω_{01} and Ω_{02} are defined in analogy to $\overline{\Omega}$ (see 2.9) but with $\overline{\sigma}$ replaced by

$$\sigma_{01} = \cos \chi\, \mathrm{Re}\, (a_0 a_1^*) \,, \quad (5.11)$$

and

$$\sigma_{02} = \mathrm{Re}\, (a_0 a_2^*) \,, \quad (5.12)$$

respectively.

By (5.7—10), supplemented by the relations (3.17) and (4.13), all relaxation coefficients of order ε which couple the heat flow and the friction pressure with mean values of the spin dependent expansion tensors (3.2) and (4.2), respectively, have been expressed by generalized Ω-integrals.

In contrast to $\sigma_0 = |a_0|^2$ whose dependence on γ^2 and χ is well known for certain model interaction potentials, little is known about a_1 and a_2. In order to get a feeling for the orders of magnitude involved we apply the guess:

$$a_1 = 0 \,, \quad a_2 = a_0 \,. \quad (5.13)$$

Then one has

$$\Omega_{01}^{(l,r)} = 0 \,, \quad \Omega_{02}^{(l,r)} = \Omega^{(l,r)} \quad (5.14)$$

where $\Omega^{(l,r)}$ contains the cross section $\sigma_0 = |a_0|^2$.

This model is somewhat similar to the non-spherical collision model introduced by Kagan and Maksimov [5] for classical rotating molecules. Note, however, that a scattering amplitude with property (5.13) still yields non-zero relaxation constants for the vector and tensor polarization. This is different from the classical KM-model.

With (5.13), the ratio of the relaxation coefficients (5.9, 10) becomes:

$$\omega_{+2}^{(71)} / \omega_{-1}^{(63)} = \frac{5}{4} \; \frac{7\, \Omega^{(4,3)} - 6\, \Omega^{(2,3)}}{7\, \Omega^{(2,2)}} \,. \quad (5.15)$$

For rigid spheres (5.15) reduces to

$$\omega_{+2}^{(71)}/\omega_{-1}^{(63)} = \frac{12}{7} \quad \text{or} \quad \omega_{+2}^{(71)}/\omega_{+2}^{(61)} = \frac{4}{7}\sqrt{\frac{7}{5}}\,. \tag{5.16}$$

§ 6. Relaxation Coefficients for Mean Values of Spin-Dependent Expansion Tensors

Now we shall consider the diagonal relaxation coefficients which are also of interest for the Senftleben-Beenakker effect. These are, in view of (5.6–8), the relaxation coefficients $\omega_{-1}^{(kk)}$ and $\omega_{+2}^{(kk)}$ with $k = 6,7$.

By neglecting the nonspherical part of the scattering amplitude and evaluating the spin traces (see I), one may relate these relaxation coefficients to CHAPMAN's collision brackets[11] $[\ldots,\ldots]_{12}$ which occur with gas-mixtures. The error involved in this "spherical" approximation is of order ε^2. One finds:

$$\begin{aligned}\omega_{-1}^{(66)} &= \frac{4}{15}\, n_0 [(V_1^2 - \tfrac{5}{2})\, V_{1\mu}, (V_1^2 - \tfrac{5}{2})\, V_{1\mu}]_{12}\,,\\ \omega_{-1}^{(77)} &= \frac{4}{21}\, n_0 [\overline{V_1}^{(3)}, \overline{V_1}^{(3)}]_{12}\,,\end{aligned} \tag{6.1}$$

$$\begin{aligned}\omega_{+2}^{(66)} &= \frac{2}{5}\, n_0 [\overline{V_1}^{(2)}, \overline{V_1}^{(2)}]_{12}\,,\\ \omega_{+2}^{(77)} &= \frac{2}{27}\, n_0 [\overline{V_1}^{(4)}, \overline{V_1}^{(4)}]_{12}\,.\end{aligned} \tag{6.2}$$

The collision brackets contain the cross section $\sigma_0 = |a_0|^2$. They are defined by

$$[\Phi_1, \Psi_1]_{12} = 8\, c_0\, \pi^{-3/2} \int \mathrm{d}^3 V\, e^{-V^2} \{\Phi_1 (\Psi_1 - \Psi_1')\, \sigma_0\} \tag{6.3}$$

with

$$\Phi_1 = \Phi(V_1)\,, \quad \Psi_1 = \Psi(V_1)\,, \quad \Psi_1' = \Psi(V_1')\,.$$

Again $\frac{\mathrm{tr}_1\, \mathrm{tr}_2}{(2S+1)^2}$ occurring with the bracket $\{\ \}$ can be replaced by 1 here. Further, in (6.2) the abbreviation

$$[\overline{V_1}^{(l)}, \overline{V_1}^{(l)}]_{12} = [\overline{V_{1\mu_1} \cdots V_{1\mu_l}}, \overline{V_{1\mu_1} \cdots V_{1\mu_l}}]_{12} \tag{6.4}$$

has been used.

The above collision brackets in turn can be expressed in terms of CHAPMAN's Ω-integrals containing the cross section σ_0. For the collision bracket occurring in (6.1) see Ref.[11]. Those occurring in (6.2) may be inferred from

$$\begin{aligned}&[\overline{V_1}^{(l)}, \overline{V_1}^{(l)}]_{12}\\ &\quad = c_0\, 2^{3-2l} (2l+1)\, l! \sum_{n=1}^{l} \binom{l}{n} \frac{2^n}{(2n+1)!!}\\ &\qquad \cdot \{\gamma^{2n} (1 - P_n(\cos\chi))\, \sigma_0\}\,.\end{aligned} \tag{6.5}$$

Here P_n denotes the Legendre polynomials with the normalization $P_n(1) = 1$. Brackets of this type have been stated by CHAPMAN and COWLING[11] for $l = 1, 2$.

Using

$$\begin{aligned}1 - P_1 &= R_1\,; \quad 1 - P_2 = \frac{3}{2} R_2\,,\\ 1 - P_3 &= \frac{5}{2} R_3 - \frac{3}{2} R_1\,,\\ 1 - P_4 &= \frac{35}{8} R_4 - \frac{15}{4} R_2\,,\end{aligned}$$

with

$$R_n(\cos\chi) = 1 - \cos^n \chi\,,$$

one finally obtains:

$$\omega_{-1}^{(66)} = n_0 \left(\frac{11}{3} \Omega^{(11)} - \frac{4}{3} \Omega^{(12)} + \frac{4}{15} \Omega^{(13)} + \frac{8}{16} \Omega^{(22)} \right), \tag{6.6}$$

$$\omega_{-1}^{(77)} = n_0 \left(2\, \Omega^{(11)} + \frac{6}{5} \Omega^{(22)} + \frac{4}{21} \Omega^{(33)} - \frac{4}{35} \Omega^{(13)} \right), \tag{6.7}$$

$$\omega_{+2}^{(66)} = n_0 \left(\frac{8}{3} \Omega^{(11)} + \frac{4}{5} \Omega^{(22)} \right), \tag{6.8}$$

$$\begin{aligned}\omega_{+2}^{(77)} = n_0 \Big(&\frac{4}{3} \Omega^{(11)} + \frac{6}{5} \Omega^{(22)} + \frac{8}{21} \Omega^{(33)} - \frac{8}{35} \Omega^{(13)}\\ &+ \frac{1}{27} \Omega^{(44)} - \frac{2}{63} \Omega^{(24)} \Big)\,.\end{aligned} \tag{6.9}$$

In order to get an idea of the relative magnitude of these relaxation coefficients we refer to the model of rigid elastic spheres (cf. Ref. 11) with diameter d. Introducing $\Omega_0 = \pi c_0 d^2$, one obtains

$$\omega_{-1}^{(66)} = \frac{59}{15}\, n_0 \Omega_0 \approx 3.94\, n_0 \Omega_0\,, \tag{6.10}$$

$$\omega_{-1}^{(77)} = \frac{186}{35}\, n_0 \Omega_0 \approx 5.31\, n_0 \Omega_0\,, \tag{6.11}$$

$$\omega_{+2}^{(66)} = \frac{64}{15}\, n_0 \Omega_0 \approx 4.26\, n_0 \Omega_0\,, \tag{6.12}$$

$$\omega_{+2}^{(77)} = \frac{1912}{315}\, n_0 \Omega_0 \approx 6.07\, n_0 \Omega_0\,. \tag{6.13}$$

These numbers should be compared with

$$\omega_{+2}^{(11)} = \frac{3}{2}\, \omega_{-1}^{(33)} = \frac{16}{5}\, n_0 \Omega_0 = 3.2\, n_0 \Omega_0\,. \tag{6.14}$$

So, $\omega_{-1}^{(66)}$ and $\omega_{+2}^{(66)}$ are almost equal to each other, but larger than the viscosity relaxation coefficient $\omega_{+2}^{(11)}$; still larger are $\omega_{-1}^{(77)}$ and $\omega_{+2}^{(77)}$.

§ 7. Concluding Remarks

In this paper, we have been mainly concerned with the order of magnitude of various collision brackets between spin-independent and spin-dependent expansion tensors. For this purpose a scaling factor ε has been introduced which essentially measures the ratio of the nonspherical and the spherical parts of the interaction potential. In particular, some of the relaxation coefficients $\omega_{-1}^{(k3)}$ and $\omega_{+2}^{(k1)}$ (see 3.1 and 4.1) which are of interest for the Senftleben-Beenakker effect on the heat conductivity and viscosity, respectively, have been considered. The results can be summarized and generalized as follows:

Since the nonspherical interaction potentials (5.1) and (5.3) contain 2nd rank irreducible spin tensors all terms of order ε of the scattering amplitude contain 2nd rank spin tensors too. This implies that all those collision brackets (3.1) and (4.1) whose spin dependent vectors and tensors are constructed from irreducible spin tensors of rank $l \neq 2$, certainly are at least of order ε^2. Only vectors quadratic in $\boldsymbol{s}$, e.g.

$$S_{3/2}^{(n)}(V^2)\,\overline{s_\mu s_\nu}\,V_\nu\,, \tag{7.1}$$

$$S_{7/2}^{(n)}(V^2)\,\overline{V_\mu V_\nu V_\lambda}\,s_\nu s_\lambda\,, \tag{7.2}$$

and tensors quadratic in $\boldsymbol{s}$, e.g.

$$S_{1/2}^{(n)}(V^2)\,\overline{s_\mu s_\nu}\,, \tag{7.3}$$

$$S_{5/2}^{(n)}(V^2)\,\overline{\overline{V_\mu V_\varrho}\,\overline{s_\varrho s_\nu}}\,, \tag{7.4}$$

$$S_{9/2}^{(n)}(V^2)\,\overline{V_\mu V_\nu V_\lambda V_\varkappa}\,\overline{s_\lambda s_\varkappa}\,, \tag{7.5}$$

lead to relaxation coefficients $\omega_{-1}^{(k3)}$ and $\omega_{+2}^{(k1)}$, respectively, which may be of order ε. Here $S_{l+\frac{1}{2}}^{(n)}$ denotes a Sonine polynomial. For the first few of these vectors and tensors see (3.2) and (4.2).

It turns out that on the one hand the Kagan vector $\Phi^{(4)}$ (of type 7.1 with $n = 0$) and all tensors of type (7.3) yield coupling relaxation coefficients of order ε^3. On the other hand, vectors of the type (7.1) with $n \geqq 1$, all vectors of type (7.2), and all tensors of type (7.4) and (7.5), indeed, lead to relaxation coefficients $\omega_{-1}^{(k3)}$ and $\omega_{+2}^{(k1)}$ respectively, which are of order ε. The first few of them have been expressed in terms of generalized Ω-integrals.

In this paper (energetically) elastic collisions have been discussed. If inelastic collisions are important, the orders of magnitude of the various coupling relaxation coefficients may be different. So, for example[7,8,13], both $\omega_{-1}^{(34)}$ and $\omega_{+2}^{(12)}$ are then of order ε^2, in contrast to (5.6). There one also has a relaxation coefficient of order ε coupling the rotational heat flux (which does not occur with spin particles) with the KAGAN vector[7,8,13].

For diatomic gases at room temperature, the KAGAN vector and the tensor polarization are most important for the SENFTLEBEN-BEENAKKER effect of the heat conductivity and the viscosity[9]. For a gas of spin particles[14], e.g. ortho-H_2 at low temperatures, the vectors $\Phi_\mu^{(6)}$, $\Phi_\mu^{(7)}$ (cf. 3.2) and the second rank tensors $\Phi_{\mu\nu}^{(6)}$, $\Phi_{\mu\nu}^{(7)}$ (cf. 4.2), however, are the more important ones.

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